Topology and its Applications 50 (1993) 129-138
North-Holland

Selection theorems under an assumption weaker than lower semi-continuity

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Received 26 March 1991
Revised 29 January 1992

Abstract

Gutev, V.G., Selection theorems under an assumption weaker than lower semi-continuity, Topology and its Applications 50 (1993) 129-138.

Every set-valued mapping satisfying an assumption weaker than lower semi-continuity admits a lower semi-continuous selection. Besides in the selection theory, this result is also successful in solving the problem of extending lower semi-continuous mappings from arbitrary to G_δ-subsets of metric spaces.

Keywords: Set-valued mapping, selection, lower semi-continuous, quasi lower semi-continuous.

AMS (MOS) Subj. Class.: Primary 54C60, 54C65; secondary 54E50, 54C20.

1. Introduction

The following two assertions might be considered as a starting point of the present paper.

Theorem 1.1 (Michael [8]). *If the mapping $\Phi : X \to \mathscr{F}(Y)$ is convex-valued and l.s.c. with X a paracompact space and Y a Banach space, then Φ admits a (single-valued) continuous selection.*

Theorem 1.2 (De Blasi and Myjak [3]). *If the mapping $\Phi : X \to \mathscr{F}(Y)$ is convex-valued and H_w-l.s.c. with X a paracompact space and Y a Banach space, then Φ admits a (single-valued) continuous selection.*

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There are examples showing that not every l.s.c. mapping is H_w-l.s.c. and vice-versa (see, e.g., [13]). So, Theorems 1.1 and 1.2 don't cover each other but nevertheless both of them are particular cases of the following more general theorem.

Theorem 1.3 (Gutev [5], Przeslawski and Rybinski [13]). *If the mapping $\Phi: X \to \mathcal{F}(Y)$ is convex-valued and q.l.s.c. with X a paracompact space and Y a Banach space, then Φ admits a (single-valued) continuous selection.*

Every l.s.c. as well as every H_w-l.s.c. mapping $\Phi: X \to \mathcal{F}(Y)$, with (Y, d) a metric space, is q.l.s.c. (see, for instance, [5] and [13]). Example 1.5 shows that there are q.l.s.c. mappings which are neither l.s.c. nor H_w-l.s.c. So, Theorem 1.3 appears to be really a "strong improvement" of both Theorem 1.1 and Theorem 1.2.

A central position in the present paper occupies the following result.

Theorem 1.4. *Every q.l.s.c. mapping $\Phi: X \to \mathcal{F}(Y)$, where X is a topological space and (Y, d) is a complete metric space, admits an l.s.c. selection.*

This result seems interesting from several points of view. In the first place, it fits naturally into the selection theory showing the best possible way of obtaining selection results for set-valued mappings of some other classes beside the class of l.s.c. ones. So, with Theorem 1.4 in mind, Theorem 1.3 is already a simple consequence of Theorem 1.1. Other results concerning set-valued "semi-continuous" selections for q.l.s.c. mappings emerge immediately if, of course, we refer to appropriate results for l.s.c. mappings (see, e.g., [2, 10]) but we shall not bore the reader by stating them. Once Theorem 1.4 is known, they follow easily and contribute nothing really new. In the second place, let us mention that Theorem 1.4 has independent interest. In Section 4, on its base, the complete metric (Čech complete metrizable) spaces are characterized between the metric (metrizable) spaces. Finally, and perhaps most important, Theorem 1.4 is a stool that can be used to deal with l.s.c. mappings in the class of metric spaces. So, in the last section, it is used in solving the problem of extending l.s.c. mappings from arbitrary to G_δ-subsets.

The proof of Theorem 1.4 is subdivided in two steps, the first of which is exhibited in Section 2 and has so little to do with q.l.s.c. mappings that it may have some general interest. The second one needs some rather preliminary considerations concerning q.l.s.c. mappings, which are found in Section 3.

Notations, definitions and examples. Let X be a topological space, (Y, d) a metric space, and let 2^Y stand for the family of nonempty subsets of Y. Set $\mathcal{F}(Y) = \{F \in 2^Y \colon F \text{ is closed}\}$. For any $F \in 2^Y$ and $\varepsilon > 0$, $B_\varepsilon(F)$ will denote the ε-neighbourhood of F, i.e., $B_\varepsilon(F) = \{y \in Y \colon d(y, F) < \varepsilon\}$. We freely use all conventional notation, such as $\bar{A}$ to denote the *closure* of an $A \subset X$, $\operatorname{int}(A)$ to denote the *interior* of an $A \subset X$, $\mathbb{R}$ to denote the real line.

A set-valued mapping $\Phi: X \to 2^Y$ is *lower semi-continuous*, or *l.s.c.*, if $\Phi^{-1}(U) = \{x \in X: \Phi(x) \cap U \neq \emptyset\}$ is open in X for every open $U \subset Y$. A set-valued mapping $\Phi: X \to 2^Y$ is *weakly Hausdorff lower semi-continuous* [3], or *H_w-l.s.c.*, if for every $x \in X$, every neighbourhood V of x and every $\varepsilon > 0$ there is a neighbourhood U of x ($U \subset V$) and a point $x' \in U$ such that $\Phi(x') \subset \bigcap\{B_\varepsilon(\Phi(z)): z \in U\}$. A set-valued mapping $\Phi: X \to 2^Y$ is *quasi lower semi-continuous* (*q.l.s.c.* for short)[1] [5, 13] if for every $x \in X$, every neighbourhood V of x, and every $\varepsilon > 0$ there exists a point $x' \in V$ such that for every point $y \in \Phi(x')$ there is a neighbourhood U_y of x for which $y \in \bigcap\{B_\varepsilon(\Phi(z)): z \in U_y\}$.

Let $\Phi: X \to 2^Y$. A set-valued mapping $\psi: X \to 2^Y$ (respectively, a single-valued mapping $f: X \to Y$) is a *selection* for Φ if $\psi(x) \subset \Phi(x)$ (respectively, $f(x) \in \Phi(x)$) for all $x \in X$.

Example 1.5 [5]. A q.l.s.c. convex-valued mapping $\Phi: [-1, 1] \to \mathscr{F}(\mathbb{R})$ which is neither l.s.c. nor H_w-l.s.c.

Proof. Define

$$\Phi(x) = \begin{cases} [-1/n, +\infty), & \text{if } x = -1/n \text{ and } n = 1, 2, \dots, \\ (-\infty, n], & \text{if } x = 1/n \text{ and } n = 1, 2, \dots, \\ \mathbb{R}, & \text{otherwise.} \end{cases}$$

In order to check that Φ is q.l.s.c., suppose that a point $x \in [-1, 1]$, a neighbourhood V of x and an $\varepsilon > 0$ are given. Some problems appear only when $x = 0$. In this case take $x' = -1/n_0$, where n_0 is a positive integer such that $-1/n_0 \in V$ and $1/n_0 < \varepsilon$. That this works, note that $y \in \Phi(x')$ implies $y \in \bigcap\{B_\varepsilon(\Phi(z)): z \in U_y\}$, where $U_y = (x', 1/n_y)$ for some integer $n_y > y$.

Now, Φ is not l.s.c. since

$$\Phi^{-1}((-\infty, -1)) = [-1, 1] \setminus \{-1/n: n = 1, 2, \dots\}$$

is not open in $[-1, 1]$; Φ is not H_w-l.s.c. because, whenever U is a neighbourhood of $0 \in [-1, 1]$ and $\varepsilon > 0$, $\Phi(x) \setminus \bigcap\{B_\varepsilon(z): z \in U\} \neq \emptyset$ for all $x \in U$. That completes the proof. □

In conclusion, a simple example showing that Theorem 1.4 becomes false if "complete" is omitted.

Example 1.6. A metric space (Y, d), which is not complete, and a q.l.s.c. mapping $\Phi: [0, 1] \to \mathscr{F}(Y)$ for which there is no l.s.c. selection.

Proof. Let $Y = (0, +\infty)$, and let d be defined by $d(y, z) = |y - z|$. The required map Φ we define as follows:

$$\Phi(x) = \begin{cases} 1/n, & \text{if } x = 1/n \text{ and } n = 1, 2, \dots, \\ Y, & \text{otherwise.} \end{cases}$$

[1] In Przeslawski and Rybinski's [13] terminology, *weakly lower semi-continuous* (*w.l.s.c.* for short).

Admitting that $\varphi : [0, 1] \to \mathscr{F}(Y)$ is an l.s.c. selection for Φ, we get that $0 \in \varphi(0)$ which is impossible. That completes the proof. □

2. Proof of Theorem 1.4

Let X be a space, (Y, d) a complete metric space and $\Phi : X \to \mathscr{F}(Y)$ a q.l.s.c. mapping. By a result of Isbell [6], there exist a strongly zero-dimensional paracompact space Z and a continuous open map g from Z onto X. It is straightforward to check that the composition $\varphi = \Phi \circ g : Z \to \mathscr{F}(Y)$ is q.l.s.c. too. In Section 3 we shall prove that φ admits a continuous selection $f : Z \to Y$ (Theorem 3.1). The set-valued mapping $f \circ g^{-1}$ is readily seen to be an l.s.c. selection for Φ.

The rest part of this section is devoted to an important refinement of Theorem 1.4.

To each mapping $\Phi : X \to \mathscr{F}(Y)$ admitting at least one l.s.c. selection we associate another one Φ_0 defined by

$$\Phi_0(x) = \bigcup\{\varphi(x) \colon \varphi \text{ is an l.s.c. selection for } \Phi\}.$$

Since an l.s.c. selection is by definition nowhere empty, $\Phi_0 : X \to 2^Y$. Moreover, it is worthwhile to observe that $x \to \overline{\Phi_0(x)}$ is an l.s.c. mapping so that $\Phi_0 : X \to \mathscr{F}(Y)$ and it is actually the *largest* l.s.c. submap of Φ.

In addition, to each mapping $\Phi : X \to \mathscr{F}(Y)$ we associate the mapping $\Phi' : X \to 2^Y \cup \{\emptyset\}$ defined by

$$\Phi'(x) = \{y \in \Phi(x) \colon x \in \operatorname{int}(\Phi^{-1}(W)) \text{ for every neighbourhood } W \text{ of } y\}$$

which is known as the derived mapping of Φ [1].

Theorem 2.1. *If the mapping $\Phi : X \to \mathscr{F}(Y)$ is q.l.s.c. with X a T_1-space and (Y, d) a complete metric space, then $\Phi' = \Phi_0$ and hence $\Phi' : X \to \mathscr{F}(Y)$ is l.s.c.*

Proof. Let $x \in X$. That $\Phi_0(x) \subset \Phi'(x)$ is obvious. Suppose $y \in \Phi'(x)$. Define a set-valued mapping $\Phi_y : X \to \mathscr{F}(Y)$ by letting $\Phi_y(x) = \{y\}$ and $\Phi_y(z) = \Phi(z)$ otherwise. Since X is a T_1-space and since $x \in U_\varepsilon = \operatorname{int}(\Phi^{-1}(B_\varepsilon(y)))$ for every $\varepsilon > 0$, it follows that Φ_y is q.l.s.c. too. Then, by Theorem 1.4, $y \in \Phi_0(x)$ and therefore $\Phi'(x) \subset \Phi_0(x)$. The theorem is proved. □

Remark. Denote $\mathscr{C}'(Y) = \mathscr{C}(Y) \cup \{Y\}$ where $\mathscr{C}(Y) = \{F \in \mathscr{F}(Y) \colon F \text{ is compact}\}$. There is a simple consequence of Theorem 2.1 dealing with set-valued mappings in the form of $\Phi : X \to \mathscr{C}'(Y)$. In order to state it, we need the following concept: We shall say that a subset M of X is *negligible with respect to* $\Phi : X \to 2^Y$, or a *Φ-negligible set*, provided $M \subset \operatorname{int}(\Phi^{-1}(U))$ for every nonempty open $U \subset Y$. Here is an example: $M = \{x \in X \colon \Phi(x) = Y\}$ is a Φ-negligible set for every l.s.c. $\Phi : X \to 2^Y$. Note now that $\Phi'|_M = \Phi|_M$ for every Φ-negligible set M. Then the following holds:

Under the assumptions of Theorem 2.1 let, in addition, $\{\Phi(x) \colon x \in X\} \subset \mathscr{C}'(Y)$ and $\{x \in X \colon \Phi(x) = Y\}$ be a Φ-negligible set. Then $\Phi' = \Phi_0$ and hence $\Phi' : X \to \mathscr{C}'(Y)$ is l.s.c.

On the base of this result and some results of [12], one can easy formulate and prove a list of selection theorems for q.l.s.c. mappings with collectionwise normal domains.

3. Q.l.s.c. mappings in paracompact spaces

As promised in the previous section, here will be proved the following theorem.

Theorem 3.1. *Let Z be a paracompact space with* $\dim(Z) \leqslant 0$, (Y, d) *a complete metric space, and let $\varphi : Z \to \mathcal{F}(Y)$ be q.l.s.c. Then φ admits a single-valued continuous selection.*

In preparation for the proof of Theorem 3.1, we begin by proving a characterization (Lemma 3.3) of q.l.s.c. mappings defined on paracompacta. First, let us point out the following alternative definition of quasi lower semi-continuity.

Proposition 3.2 [13]. *Let Z be a topological space, (Y, d) a metric space, and $\varphi : Z \to 2^Y$. Then φ is q.l.s.c. if and only if for every $\varepsilon > 0$, every $z \in Z$, every neighbourhood V of z, and every (not necessarily continuous) selection $h : Z \to Y$ for φ there exists a neighbourhood U $(U \subset V)$ of z such that $h(V) \cap (\bigcap\{B_\varepsilon(\varphi(x)) : x \in U\}) \neq \emptyset$.*

Proof. By contradiction. □

Let ν and γ be collections of subsets of Z. A map $r : \nu \to \gamma$ is called *refining* [5] provided $W \subset r(W)$ for all $W \in \nu$.

Lemma 3.3. *Let Z be a paracompact space, (Y, d) a metric space, and $\varphi : Z \to 2^Y$. Then φ is q.l.s.c. if and only if for every $\varepsilon > 0$, every locally finite open cover γ of Z, and every (not necessarily continuous) selection $h : Z \to Y$ for φ there exists a locally finite open cover ν of Z and a refining map $r : \nu \to \gamma$ such that, for every $W \in \nu$,*

$$h(r(W)) \cap (\bigcap\{B_\varepsilon(\varphi(x)) : x \in W\}) \neq \emptyset.$$

Proof. Sufficiency follows immediately by Proposition 3.2.

Necessity. Suppose $\varphi : Z \to 2^Y$ is q.l.s.c. Let, moreover, an $\varepsilon > 0$, a locally finite open cover γ of Z, and a selection $h : Z \to Y$ for φ be given; we look for a locally finite open cover ν of Z and a refining map $r : \nu \to \gamma$ such that

$$h(r(W)) \cap (\bigcap\{B_\varepsilon(\varphi(x)) : x \in W\}) \neq \emptyset \quad \text{for every } W \in \nu.$$

Define a map $s : Z \to \gamma$ such that $z \in s(z)$ for each $z \in Z$. Then, for every point $z \in Z$, by Proposition 3.2 with $V = s(z)$, we find a neighbourhood U_z of z such that

$$U_z \subset s(z) \quad \text{and} \quad h(s(z)) \cap (\bigcap\{B_\varepsilon(\varphi(x)) : x \in U_z\}) \neq \emptyset.$$

Because of the paracompactness of Z, there now exists a locally finite open cover ν of Z which refines $\{U_z : z \in Z\}$. This is our ν. To define $r : \nu \to \gamma$, for each $W \in \nu$, pick a fixed point $z(W) \in Z$ such that $W \subset U_{z(W)}$, and then merely set $r(W) = s(z(W))$. That completes the proof. □

Returning now to Theorem 3.1, let us mention that its proof follows Michael's scheme from [11]. The key step in that proof, as in the proofs of almost all selection theorems (going back at least to Michael [8]), is the construction of "continuous ε-approximate" selections for φ. This is done as follows:

Lemma 3.4. *In the conditions of Theorem* 3.1 *let, in addition,* $\varepsilon > 0$, γ *be a locally finite open cover of* Z *and let* $h: Z \to Y$ *be a selection for* φ. *Then there exist a locally finite open cover* ω *of* Z, *a refining map* $t: \omega \to \gamma$ *and a continuous map* $f: Z \to Y$ *such that* $f(z) \in h(t(U)) \cap B_\varepsilon(\varphi(z))$ *for every* $z \in U \in \omega$.

Proof. Using Lemma 3.3, we find a locally finite open cover ν of Z and a refining map $r: \nu \to \gamma$ such that, for every $W \in \nu$,

$$h(r(W)) \cap (\bigcap\{B_\varepsilon(\varphi(x)) : x \in W\}) \neq \emptyset.$$

Since $\dim(Z) \leq 0$ and since ν is a locally finite open cover, by [4], there now is an open disjoint cover ω of Z which refines ν. This is our ω. Next, for every $U \in \omega$, pick a fixed $W_u \in \nu$ with $U \subset W_u$, and then define $t: \omega \to \gamma$ by $t(U) = r(W_u)$. As for the map $f: Z \to Y$, we first define maps $s: Z \to \omega$ and $g: \omega \to Y$ such that

$$z \in s(z) \quad \text{and} \quad g(U) \in h(t(U)) \cap (\bigcap\{B_\varepsilon(\varphi(x)) : x \in U\})$$

whenever $z \in Z$ and $U \in \omega$. Then our f is the composition: $f = g \circ s$. That completes the proof. □

First, immediately from Lemma 3.4, we get the following

Corollary 3.5. *Under the assumptions of Theorem* 3.1, *there is a continuous* $f_1: Z \to Y$ *such that* $f_1(z) \in B_{2^{-1}}(\varphi(z))$ *for all* $z \in Z$.

Next, by the same lemma, we establish also another one

Corollary 3.6. *In the conditions of Theorem* 3.1 *let, in addition,* $f_n: Z \to Y$ *be a continuous map such that* $f_n(z) \in B_{2^{-n}}(\varphi(z))$ *for all* $z \in Z$. *Then there is a continuous map* $f_{n+1}: Z \to Y$ *such that*

$$f_{n+1}(z) \in B_{2^{-(n+1)}}(\varphi(z)) \cap B_{2^{-(n-1)}}(f_n(z)) \quad \textit{for all } z \in Z.$$

Proof. For every $z \in Z$ fix a point $h(z) \in B_{2^{-n}}(f_n(z)) \cap \varphi(z)$. Next, take a locally finite open cover γ of Z such that the diameter of $f_n(V) < 2^{-n}$ for all $V \in \gamma$; such a γ exists because of the continuity of f_n. Now, by Lemma 3.4 with $\varepsilon = 2^{-(n+1)}$, we get a locally finite open cover ω of Z, a refining map $t: \omega \to \gamma$ and a continuous $f_{n+1}: Z \to Y$ such that

$$f_{n+1}(z) \in h(t(U)) \cap B_{2^{-(n+1)}}(\varphi(z)) \quad \text{for every } z \in U \in \omega.$$

It only remains to verify that $f_{n+1}(z) \in B_{2^{-(n-1)}}(f_n(z))$ for all $z \in Z$. Take a point $z \in Z$ and a set $U \in \omega$ containing z. Then $h(t(U)) \subset B_{2^{-n}}(f_n(t(U)))$ and the diameter of $f_n(t(U)) < 2^{-n}$ imply $f_{n+1}(z) \in h(t(U)) \subset B_{2^{-(n-1)}}(f_n(z))$. That completes the proof. □

Proof of Theorem 3.1. By induction, using Corollaries 3.5 and 3.6, we construct a sequence $\{f_n\}$ of continuous maps $f_n : Z \to Y$ such that, for all $z \in Z$,

(i) $f_n(z) \in B_{2^{-n}}(\varphi(z)), \qquad n = 1, 2, \ldots,$

(ii) $f_{n+1}(z) \in B_{2^{-(n-1)}}(f_n(z)), \qquad n = 1, 2, \ldots.$

By (ii), the sequence $\{f_n\}$ is uniformly Cauchy, so it must converge to some continuous $f : Z \to Y$. By (i), $f(z) \in \varphi(z)$ for all $z \in Z$. Thus f is a single-valued continuous selection for φ. That completes the proof. □

4. Applications to complete metric spaces and l.s.c. mappings between them

Let us begin with two constructions of new q.l.s.c. mappings starting with given ones.

Proposition 4.1. *Let (X, p) and (Y, d) be metric spaces, $A \subset X$, and $f : A \to Y$ be uniformly continuous. Define a set-valued mapping $\Phi : X \to \mathcal{F}(Y)$ by $\Phi(x) = \{f(x)\}$ for $x \in A$ and $\Phi(x) = Y$ otherwise. Then Φ is q.l.s.c.*

Proof. Routine verification. □

Proposition 4.2. *Let (Z, p) and (Y, d) be metric spaces, $A \subset Z$, and $\varphi : A \to \mathcal{C}(Y)$ be l.s.c. Define a set-valued mapping $\Phi : Z \to \mathcal{F}(Y)$ by $\Phi(z) = \varphi(z)$ for $z \in A$ and $\Phi(z) = Y$ otherwise. Then there exists a G_δ-set $X \subset Z$ containing A and such that $\Phi|_X$ is q.l.s.c.*

Proof. Since φ is an l.s.c. compact-valued mapping, for each $a \in A$ and each integer $n > 0$, we can fix a neighbourhood $V_n(a)$ of a in Z such that the diameter of $V_n(a)$ $< 1/n$ and $\varphi(a) \subset B_{1/n}(\varphi(z))$ for all $z \in V_n(a) \cap A$ (see [9, Lemma 11.3]). Let $U_n = \bigcup\{V_n(a) \colon a \in A\}$. Our X is now the intersection $\bar{A} \cap (\bigcap\{U_n \colon n = 1, 2, \ldots\})$. In order to check that $\Phi|_X$ is q.l.s.c., suppose $x \in X$, V is a neighbourhood of x and $\varepsilon > 0$. Pick a fixed $n(\varepsilon) > 0$ and a point $x' \in V \cap A$ such that $1/n(\varepsilon) < \varepsilon$ and $x \in V_{n(\varepsilon)}(x')$. Then $y \in \Phi(x')$ implies $x \in V_{n(\varepsilon)}(x') \subset \Phi^{-1}(B_\varepsilon(y))$. That is $y \in \bigcap\{B_\varepsilon(\Phi(z)) \colon z \in U_y\}$ for $U_y = V_{n(\varepsilon)}(x')$ which completes the proof. □

These constructions allows one to establish the following two characterizations of the completeness in metric spaces. First, let us recall that a set-valued mapping $\varphi : X \to 2^Y$ is *upper semi-continuous*, or *u.s.c.*, if $\varphi^{\#}(U) = \{x \in X \colon \varphi(x) \subset U\}$ is open in X for every open $U \subset Y$.

Theorem 4.3. *Let (Y, d) be a metric space. Then the following conditions are equivalent*:

(a) *(Y, d) is a complete metric space*;

(b) *for every metrizable space X, every q.l.s.c. $\Phi : X \to \mathcal{F}(Y)$ has an l.s.c. selection*;

(c) *for every metrizable space X, every q.l.s.c. $\Phi : X \to \mathcal{F}(Y)$ has a u.s.c. selection.*

Proof. That (a)$\to$(b), it follows immediately from Theorem 1.4; as for (a)$\to$(c), use first Theorem 2.1 and then apply [10, Theorem 1.1] to the map Φ'.

(b)$\to$(a) (respectively, (c)$\to$(a)). Let $(\tilde{Y}, d)$ be the completion of (Y, d). Define a set-valued mapping $\Phi: \tilde{Y} \to \mathscr{F}(Y)$ by letting $\Phi(y) = \{y\}$ if $y \in Y$ and $\Phi(y) = Y$ otherwise. By Proposition 4.1, Φ is q.l.s.c. Then, by our assumption, there is an l.s.c. (respectively, a u.s.c.) selection $\varphi: \tilde{Y} \to 2^Y$ for Φ. Observe that φ is an l.s.c. (repectively, a u.s.c.) retraction, i.e., $\varphi(y) = \{y\}$ for all $y \in Y$. Hence $\tilde{Y} = Y$, because $\{y \in \tilde{Y}: \varphi(y) \text{ is a singleton}\} = \{y \in \tilde{Y}: \varphi(y) = \{y\}\} = Y$ is closed in $\tilde{Y}$. That completes the proof. $\square$

Theorem 4.4. *For a metrizable space Y, the following two conditions are equivalent*:

(a) *Y is Čech complete*;

(b) *whenever Z is metrizable and $A \subset Z$, every l.s.c. $\varphi: A \to \mathscr{C}(Y)$ can be extended to an l.s.c. mapping $\tilde{\varphi}$ from some G_δ-set $X \supset A$ to $\mathscr{F}(Y)$.*

Proof. (a)$\to$(b). Let Z, A and φ be as in (b). Let, moreover, p be a metric on Z and d be a complete metric on Y. Define a set-valued mapping $\Phi: Z \to \mathscr{F}(Y)$ by letting $\Phi(z) = \varphi(z)$ for $z \in A$ and $\Phi(z) = Y$ otherwise. By virtue of Proposition 4.2, there is a G_δ-set $X \supset A$ such that $\Phi|_X$ is q.l.s.c. Take now $\tilde{\varphi} = (\Phi|_X)'$. By Theorem 2.1, $\tilde{\varphi}: X \to \mathscr{F}(Y)$ is l.s.c. That $\tilde{\varphi}$ is an extension of the mapping φ is obvious.

(b)$\to$(c). Let d be a metric on Y, and let $(\tilde{Y}, d)$ be the completion of (Y, d). By (b) with $Z = \tilde{Y}$ and $A = Y$, the identical map $\varphi: Y \to Y$ can be extended to an l.s.c. $\tilde{\varphi}: X \to \mathscr{F}(Y)$ from some G_δ-set $X \supset Y$. Then $Y = X$ because $\tilde{\varphi}$ is an l.s.c. retraction, which completes the proof. $\square$

In conclusion, using Theorem 4.4, we shall obtain a set-valued version (Theorem 4.7) of Lavrentieff's theorem [7] on extensions of continuous maps to complete metric spaces.

For a space Y, denote $\mathscr{F}_s(Y) = \{F \in \mathscr{F}(Y): F \text{ is separable}\}$. We use also $|S|$ to denote the cardinality of any set S.

Lemma 4.5. *Let Y be a complete matrizable space, A a metrizable space, and let $\Phi: A \to \mathscr{F}_s(Y)$ be l.s.c. Then for every locally-finite collection γ of open subsets of Y there exists a countable collection $\mathscr{T}(\gamma)$ of l.s.c. selections $\varphi: A \to \mathscr{C}(Y)$ for Φ with the following property*:

For every $a \in A$ and every $V \in \gamma$, with $\Phi(a) \cap V \neq \emptyset$, there is a $\varphi \in \mathscr{T}(\gamma)$ such that $\varphi(a) \subset \overline{\Phi(a) \cap V}$.

Proof. Let $\mathscr{Z} = \{(a, V) \in A \times \gamma: \Phi(a) \cap V \neq \emptyset\}$, and let $\alpha: \mathscr{Z} \to A$ and $\delta: \mathscr{Z} \to \gamma$ be the projections onto A and, respectively, onto γ. Let also $\mathscr{B} = \bigcup\{\mathscr{B}_n: n = 1, 2, \ldots\}$ be a σ-discrete base of A (i.e., each $\mathscr{B}_n$ is a discrete collection in A). Pick, for every $z \in \mathscr{Z}$, a fixed $B_z \in \mathscr{B}$ such that

(i) $\alpha(z) \in B_z \subset \overline{B_z} \subset \Phi^{-1}(\delta(z))$, and then define $\beta:\mathscr{X}\to\mathscr{B}$ by $\beta(z)=B_z$. Note that such a B_z certainly exists because $\Phi^{-1}(\delta(z))$ is a neighbourhood of $\alpha(z)$ (Φ is l.s.c.) and because $\mathscr{B}$ is a base of A. Now, for every n, set $\mathscr{X}_n=\beta^{-1}(\mathscr{B}_n)$ and $\mathscr{G}_n=\beta(\mathscr{X}_n)$. The following holds:

(ii) $|\delta(\beta^{-1}(B))|\leq\aleph_0$ for every $B\in\mathscr{G}_n$.

Indeed, let $z_0\in\beta^{-1}(B)$. Then $z\in\beta^{-1}(B)$ implies $B_z=B=B_{z_0}$, and therefore, by (i), $\alpha(z_0)\in B_z\subset\Phi^{-1}(\delta(z))$. Hence, $\Phi(\alpha(z_0))\cap\delta(z)\neq\emptyset$. That is, $\delta(\beta^{-1}(B))\subset\{V\in\gamma\colon \Phi(\alpha(z_0))\cap V\neq\emptyset\}$. Since now $\Phi(\alpha(z_0))$ is separable and γ is locally finite, $|\{V\in\gamma\colon \Phi(\alpha(z_0))\cap V\neq\emptyset\}|\leq\aleph_0$ which, in effect, is (ii).

Fix now a countable set, say the set of natural numbers $\mathbb{N}$. By virtue of (ii), for every $B\in\mathscr{G}_n$, there is a map f_B from $\mathbb{N}$ onto $\delta(\beta^{-1}(B))$. Next, for every $k\in\mathbb{N}$, define a set-valued mapping $\Phi_k^n\colon A\to\mathscr{F}(Y)$ by $\Phi_k^n(a)=\overline{\Phi(a)\cap f_B(k)}$ if $a\in\bar{B}$ for some $B\in\mathscr{G}_n$ and $\Phi_k^n(a)=\Phi(a)$ otherwise. Since $\{\bar{B}\colon B\in\mathscr{G}_n\}$ is discrete ($\mathscr{G}_n\subset\mathscr{B}_n$), Φ_k^n is well defined. In addition, Φ_k^n is l.s.c. because $\Phi_k^n|_{\bar{B}}$ is an l.s.c. selection for $\Phi|_{\bar{B}}$ (see [8, Propositions 2.3 and 2.4]). Then by [10, Theorem 1.1], each Φ_k^n admits an l.s.c. selection $\varphi_k^n\colon A\to\mathscr{C}(Y)$. Setting finally $\mathscr{T}(\gamma)=\{\varphi_k^n\colon n,k=1,2,\ldots\}$, it only remains to verify that this works. Let $a\in A$, $V\in\gamma$ and let $\Phi(a)\cap V\neq\emptyset$. Then $(a,V)\in\mathscr{X}$. Since now $z=(a,V)\in\mathscr{X}_n$ for some n, there is a $B=B_z\in\mathscr{G}_n$ for which $a\in B$ and $V=\delta(z)\in\delta(\beta^{-1}(B))$. Hence, there is a $k\in\mathbb{N}$ such that $f_B(k)=V$. Then

$$\varphi_k^n(\alpha)\subset\Phi_k^n(a)=\overline{\Phi(a)\cap f_B(k)}=\overline{\Phi(a)\cap V},$$

which completes the proof. □

Lemma 4.6. *Let Y be a complete metrizable space, A a metrizable space, and let $\Phi\colon A\to\mathscr{F}_s(Y)$ be l.s.c. Then there exists a countable collection $\mathscr{T}$ of l.s.c. selections $\varphi\colon A\to\mathscr{C}(Y)$ for Φ such that, for every $a\in A$, $\bigcup\{\varphi(a)\colon\varphi\in\mathscr{T}\}$ is dense in $\Phi(a)$.*

Proof. Let $\{\gamma_n\colon n=1,2,\ldots\}$ be a sequence of open and locally-finite covers of Y such that the diameter of $V<2^{-n}$ for every $V\in\gamma_n$. For each n, let $\mathscr{T}(\gamma_n)$ be as in Lemma 4.5 applied to the cover γ_n. Then $\mathscr{T}=\bigcup\{\mathscr{T}(\gamma_n)\colon n=1,2,\ldots\}$ is our. That completes the proof. □

Theorem 4.7. *Let Y be a complete metrizable space, Z a metrizable space, $A\subset Z$, and let $\Phi\colon A\to\mathscr{F}_s(Y)$ be l.s.c. Then Φ can be extended to an l.s.c. map $\tilde{\Phi}\colon X\to\mathscr{F}(Y)$ for some G_δ-set $X\supset A$.*

Proof. By virtue of Lemma 4.6, there is a countable collection $\mathscr{T}$ of l.s.c. selections $\varphi\colon A\to\mathscr{C}(Y)$ for Φ such that, for every $a\in A$, $\bigcup\{\varphi(a)\colon\varphi\in\mathscr{T}\}$ is dense in $\Phi(a)$. By Theorem 4.4, for each $\varphi\in\mathscr{T}$, there is a G_δ-set $X_\varphi\supset A$ and an l.s.c. mapping $\tilde{\varphi}\colon X_\varphi\to\mathscr{F}(Y)$ such that $\tilde{\varphi}|_A=\varphi$. Set $X=\bigcap\{X_\varphi\colon\varphi\in\mathscr{T}\}$ and then define $\tilde{\Phi}\colon X\to\mathscr{F}(Y)$ by $\tilde{\Phi}(x)=\overline{\bigcup\{\varphi(x)\colon\varphi\in\mathscr{T}\}}$ for all $x\in X$. These X and $\tilde{\Phi}$ satisfy all our requirements. That completes the proof. □

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